Prosimy cytować jako: Inż. Ap. Chem. 2009, 48, 6, 168-171

TEODOR SKIEPKO

Faculty of Mechanical Engineering, Bialystok Technical University, Bialystok

The *Reynolds* transport theorem for three phase systems with interface storage

Introduction

The *Reynolds* Transport Theorem (since abbreviated RTT) is a kinematic relation expressing the accumulation rate of an Extensive Quantity (*EQ*) in a material system Σ given by Lagrangian description in terms referenced to a domain of spatially prescribed configuration (*Eulerian* description) of volume *V* (fixed or movable). The first *RTT* for heterogeneous systems of negligible interface storage is by *Truesdell* and *Toupin* [1]. If, however, contribution of the interface storage into system storage can be essential *Slattery* [2] proposed the *RTT* as

$$\frac{\mathrm{d}}{\mathrm{d}t}=\left(\int_{\Omega_\Sigma}\eta \mathrm{d}V+\int_{\Gamma_\Sigma}\eta_S \mathrm{d}A\right)=\int_V\left(\frac{D\eta}{\mathrm{D}t}+\eta\,\mathrm{div}\boldsymbol{V}\right)\mathrm{d}V+$$
$$+\int_S\left(\frac{D_S\eta_S}{Dt}+\eta_S\mathrm{div}_S\boldsymbol{V}_S+\left\|\eta(\boldsymbol{V}-\boldsymbol{U})\bullet\zeta\right\|\right)\mathrm{d}A \qquad (1)$$

where *V* stands for an established fixed volume of reference (Eulerian description), *S* is the overall interface area within *V*. $\boldsymbol{U}$ and $\boldsymbol{V_s}$ are the spatial velocities of the interfaces and the surface systems moving in, respectively. $D\eta/Dt$ and $\mathrm{D_S}\eta_S/Dt$ stand for the material derivatives of spatial and surface densities η and η_s, respectively. Term $\|\eta(\boldsymbol{V}-\boldsymbol{U})\bullet\zeta\|$ of Eq.(1) reads as $\left\|\eta^+(\boldsymbol{V}^+-\boldsymbol{U})\bullet\zeta^+ + \eta^-(\boldsymbol{V}^- -\boldsymbol{U})\bullet\zeta^-\right\|$ where ζ means the outward unit normal vector to the interface pointing into phase moving at *V* and refers to the jump condition for phasic spatial density η across the interface set between two phases which properties are denoted by superscripts + and –. Single integral symbols ∫ used throughout the paper refer to either the volume (differential d*V*), surface (d*A*) or line (d*l*) integrals, respectively.

This paper is purposed to derive the *RTT* relation for three phase systems of essential interface storage in terms referenced to control volume *CV* surrounded by control surface *CS* being in arbitrary motion with respect to fixed (inertial) reference frame.

RTT for three phase systems

In Fig. 1 a three phase material system Σ is displayed occupying spatial domain Ω of boundary Γ split into phasic portions Ω_i (*i* = 1, 2, 3). The system is composed of three spatial subsystems $\bigcup_i^3\Sigma_{vi}$ separated by interfaces *S* and *K* surface subsystems $\bigcup_k^K\Sigma_{sk}$ dwelling in *S*. System passes through a movable *CV* bounded by *CS*, see in Fig. 2a. In view of Σ is composed of spatial and surface subsystems the *CV* comprises both spatial phasic domains of total volume *V* and interfacial domains of aggregated area *S*, hence $CV = V \cup S$.

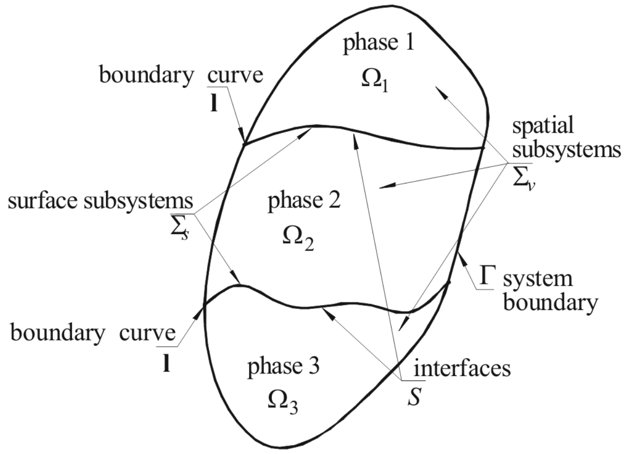


Fig. 1. System Σ composed of spatial and surface subsystems

Volume *V* involves all the phasic volumes embedded in *CV*, hence $V=\bigcup_i^3 V_i$. The aggregated interfacial area *S* involves the interfaces placed within *CV* so that $S=\bigcup_k^K S_k$. The entire *CS* consists in $CS = R \cup C$ where *R* is the aggregated external boundary of phasic domains determined as $R=\bigcup_i^3 R_i$ with understanding that each R_i is the entire external boundary accompanying the i^{th} phasic volume. $C=\bigcup_k^K C_k$ stands for the aggregated boundary curve of all individual boundary curves C_k formed as intersection of *CS* and interface S_k.

Accumulation $\delta\Phi_\Sigma$ of an extensive quantity (abbreviated *EQ*) in system Σ is determined by the difference in system storages Φ_Σ at *t* + δ*t* and *t*, hence

$$\underbrace{\delta\Phi_\Sigma}_{\text{accumulation in }\Sigma\text{ during }\delta t}=\underbrace{\Phi_\Sigma(t+\delta t)}_{\text{storage in }\Sigma\text{ at }t+\delta t}-\underbrace{\Phi_\Sigma(t)}_{\text{storage in }\Sigma\text{ at }t} \qquad (2)$$

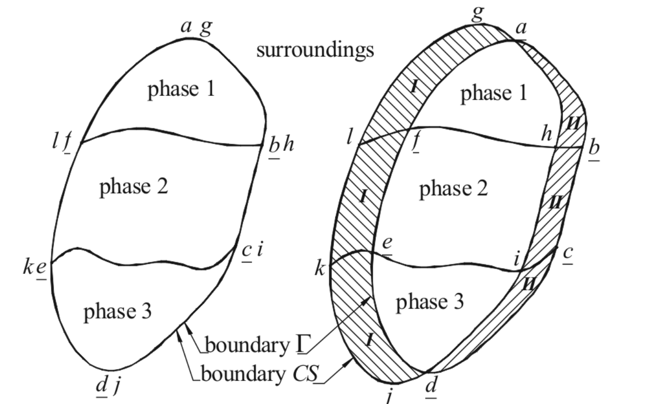
In Fig. 2a the coincidence of Ω and *CV* is shown at an instant *t*. In such particular circumstances boundary Γ of system Σ traced by lowercase letters <u>*abcdefa*</u> is superimposed upon boundary *CS* of *CV* indicated by *ghijklg*, hence <u>*abcdefa*</u> = *ghijklg*, (Fig. 2a). In turn amounts of *EQ* stored within Σ and *CV* are the same, what gives

$$\underbrace{\Phi_\Sigma(t)}_{\text{storage in }\Sigma\text{ at }t}=\underbrace{\Phi_{CV}(t)}_{\text{storage in }CV\text{ at }t} \qquad (3)$$

At instant *t* + δ*t* system Σ is displaced partially out of *CV*. Hence, boundaries of Σ traced along <u>*abcdefa*</u> and *CV* marked as *ghijklg* are shifted each other (Fig. 2b).

In turn, system Σ leaves to *CV* some amount of *EQ* stored in region ***I*** (<u>*afedjklga*</u>) and carries out of *CV* some amount of *EQ* stored in region ***II*** (<u>*abcdiha*</u>). Hence, based on Fig. 2b one gets storage $\Phi_\Sigma(t+\delta t)$ expressed in terms referenced to *CV* as

Prosimy cytować jako: Inż. Ap. Chem. 2009, 48, 6, 168-171



(a) coincidence of Σ and *CV* at *t* (b) displacement of Σ and *CV* at *t*+δ*t*

Fig. 2. System Σ passing through a moving *CV*: a) coinciding condition at time *t*; b) displacement of Σ with respect to *CV* at time *t* + δ*t*

$$\underbrace{\Phi_\Sigma(t+\delta t)}_{\substack{\text{storage in }\Sigma\\ \text{at } t+\delta t}} = \underbrace{\Phi_{CV}(t+\delta t)}_{\substack{\text{storage in } CV\\ \text{at } t+\delta t}} + \underbrace{\delta\Phi_{II}(\delta t)}_{\substack{\text{amount of EQ carried by}\\ \Sigma \text{ out of } CV \text{ during } \delta t\\ \text{(region } II)}} - \underbrace{\delta\Phi_{I}(\delta t)}_{\substack{\text{amount of EQ brought}\\ \text{in } CV \text{ by } \Sigma \text{ during } \delta t\\ \text{(region } I)}} \quad (4)$$

where $\delta\Phi_I(\delta t)$ stands for inflow of *EQ* into *CV* across *CS*. Term $\delta\Phi_{II}(\delta t)$ of Eq. (5) means the efflux of *EQ* across *CS* out of *CV*. By substitution Eqs (3, 4) into Eq. (2) one obtains

$$\underbrace{\delta\Phi_\Sigma}_{\substack{\text{accumulation in }\Sigma\\ \text{during } \delta t}} = \underbrace{\Phi_{CV}(t+\delta t)-\Phi_{CV}(t)}_{\text{accumulation in moving } CV} + \underbrace{\delta\Phi_{II}(\delta t)-\delta\Phi_I(\delta t)}_{\text{transport across } CS} \quad (5)$$

Storages $\Phi_{CV}(t)$ and $\Phi_{CV}(t+\delta t)$ of Eq.(5) include those in the spatial domains of volume V (denoted by Φ_V) and those in the interfacial domains of area S (by Φ_S). Likewise, amounts of *EQ* transported by moving Σ refer to contributions made by macroscopic movements of both spatial (by $\delta_v\Phi$) and surface (by $\delta_s\Phi$) subsystems. With this understanding, corresponding terms are introduced into Eq. (5) and subsequently all the terms on both sides are divided by δ*t*. Then by letting $\delta t \to 0$ one gets Eq. (5) expressed on the rate basis as

$$\lim_{\delta t\to 0}\frac{\delta\Phi_\Sigma}{\delta t} = \lim_{\delta t\to 0}\Bigg[\underbrace{\frac{\Phi_V(t+\delta t)-\Phi_V(t)}{\delta t}}_{\substack{\text{accumulation rate in}\\ \text{spatial phasic domains}\\ \text{of volume } V\\ (1)}} + \underbrace{\frac{\delta_V\Phi_{II}(\delta t)-\delta_V\Phi_I(\delta t)}{\delta t}}_{\substack{\text{transport rate of } EQ \text{ by}\\ \text{movement of spacial}\\ \text{subsystems across CS}\\ (2)}} +$$

$$+\underbrace{\frac{\Phi_S(t+\delta t)-\Phi_S(t)}{\delta t}}_{\substack{\text{accumulation rate in}\\ \text{interfacial domains}\\ \text{of area } S\\ (3)}} + \underbrace{\frac{\delta_S\Phi_{II}(\delta t)-\delta_S\Phi_I(\delta t)}{\delta t}}_{\substack{\text{transport rate of } EQ \text{ by}\\ \text{movement of surface}\\ \text{subsystems across CS}\\ (4)}}\Bigg] \quad (6)$$

The left side of Eq. (6) converges at the accumulation rate of *EQ* within Σ to be given by

$$\lim_{\delta t\to 0}\frac{\delta\Phi_\Sigma}{\delta t} = \frac{d\Phi_\Sigma}{dt} \quad (7)$$

Below, the right side terms of Eq. (6) are converted into rate forms referenced to moving *CV*.

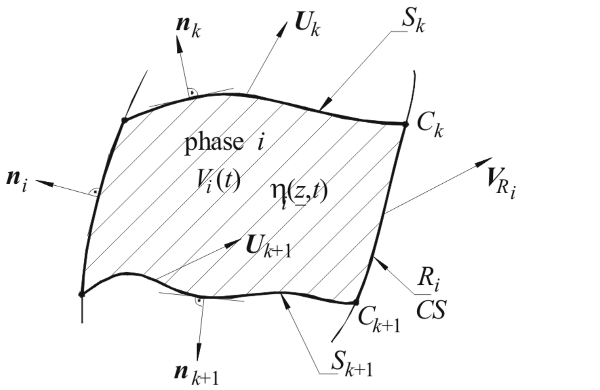


Fig. 3. Phasic domain $V_i(t)$ bounded by boundary R_i and moving interfaces S_k and S_{k+1}

Term (1)

Taking the limit the rate form of the first term of Eq. (6) becomes

$$\lim_{\delta t\to 0}\frac{\Phi_V(t+\delta t)-\Phi_V(t)}{\delta t} = \frac{\mathrm{d}\Phi_V(t)}{\mathrm{d}t} \quad (8)$$

where:

$$\frac{\mathrm{d}\Phi_V(t)}{\mathrm{d}t} = \Sigma_i^3\frac{\mathrm{d}\Phi_{V_i}(t)}{\mathrm{d}t} \quad \text{and} \quad \Phi_{V_i}(t) = \int_{V_i(t)}\eta_i(\underline{z},t)\mathrm{d}V \quad (9)$$

is the storage within the i^{th} phasic domain of moving volume $V_i(t)$ at an instant t and $\underline{z}$ are the spatial coordinates. In Fig. 3 boundaries of volume $V_i(t)$ are illustrated.

It is seen in Fig. 3 that the entire boundary $\Re_i$ of V_i is a closed surface $\Re_i = R_i + S_i$ where R_i is the external part of $\Re_i$ and $S_i = \Sigma_k^{K_i} S_k$ is the interfacial part of $\Re_i$ assembled of K_i interfaces S_k associated phase *i*. Now the generalized transport theorem [1] is applied to determine each derivative of as

$$\frac{\mathrm{d}\Phi_{V_i}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_i(t)}\eta_i(\underline{z},t)\mathrm{d}V =$$

$$\int_{V_i}\frac{\partial\eta_\iota}{\delta\tau}dV + \int_{R_i}\eta_i \boldsymbol{V}_{R_i}\bullet\boldsymbol{n}_i\mathrm{d}A + \Sigma_k^{K_i}\int_{S_k}\eta_{i\triangleright S_k}\boldsymbol{U}_k\bullet\boldsymbol{n}_k\mathrm{d}A \quad (10)$$

where boundaries R_i and S_k (Fig. 3) are moving at velocities $\boldsymbol{V}_{R_i}$ and U_k, respectively. $\boldsymbol{n}_i$ and $\boldsymbol{n}_k$ are the unit normal vectors to boundaries R_i and S_k, (k = 1, 2), respectively, drawn outward with respect to $V_i(t)$. $\eta_{i\triangleright S_k}$ is the spatial density of *EQ* stored in the i^{th} phase taken at infinitesimally close position to interface S_k. Eq. (10) can be can be modified by the use of the *Gauss's* theorem [3]. Thus one obtains

$$\int_{R_i}\eta_i\boldsymbol{V}_{R_i}\bullet\boldsymbol{n}_i\mathrm{d}A + \Sigma_k^{K_i}\int_{S_k}\eta_{i\triangleright S_k}\boldsymbol{U}_k\bullet\boldsymbol{n}_k\mathrm{d}A = \int_{V_i}\mathrm{div}(\eta_i\boldsymbol{V}_{\Re_i})\mathrm{d}V \quad (11)$$

By substitution Eq. (11) into Eq. (10) and subsequently Eq. (10) into Eq. (9) one gets

$$\frac{\mathrm{d}\Phi_V(t)}{\mathrm{d}t} = \sum_i^3\int_{V_i}\frac{\partial\eta_i}{\partial t}dV + \Sigma_i^3\int_{V_i}\mathrm{div}(\eta_i\boldsymbol{V}_{\Re_i})\mathrm{d}V =$$

$$= \int_V\frac{\partial\eta}{\partial t}dV + \int_V\mathrm{div}(\eta\boldsymbol{V}_\Re)\mathrm{d}V \quad (12)$$

Term (2)

The rate of accumulation in system by moving spatial subsystems relative to the *CV* refers to phases engaged in spatial regions ***I*** and ***II***. Hence Term (2) of Eq. (6) is

Prosimy cytować jako: Inż. Ap. Chem. 2009, 48, 6, 168-171

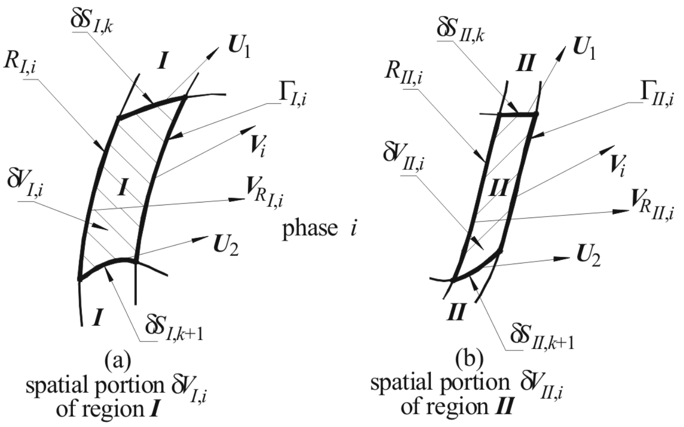


Fig. 4. The spatial portions of region *I* of volume $\delta V_{I,i}$ and region *II* of volume $\delta V_{II,i}$ accompanied by associated boundaries

$$\lim_{\delta t\to 0}\frac{\delta_V\Phi_{II}(\delta t)-\delta_V\Phi_I(\delta t)}{\delta t}=\lim_{\delta t\to 0}\frac{\sum_i^3 \delta t\frac{d\Phi_{II,i}}{dt}-\sum_i^3 \delta t\frac{d\Phi_{I,i}}{dt}}{\delta t} \quad (13)$$

Below each derivative $d\Phi_{I,i}/dt$ of Eq. (13) referenced to the i^{th} spatial portion of region ***I*** of volume $\delta V_{I,i}$ bounded by $\Gamma_{I,i}\cup R_{I,i}\cup\delta S_{I,i}$, (Fig. 4a), is determined by the generalized transport theorem [1]. Hence one gets

$$\frac{d\Phi_{I,i}}{dt}=\int_{\delta V_{I,i}}\frac{\partial\eta_i}{\partial t}dV-\int_{\Gamma_{I,i}}\eta_i\boldsymbol{V}_i\bullet\boldsymbol{n}_i dA+$$
$$+\int_{R_{I,i}}\eta_i\boldsymbol{V}_{R_i}\bullet\boldsymbol{n}_i dA+\sum_k^{K_i}\int_{\delta S_{I,k}}\eta_{i\triangleright\delta S_{I,k}}\boldsymbol{U}_k\bullet\boldsymbol{n}_k dA \quad (14)$$

where $\delta S_{I,k}$ is the k^{th} portion of $\delta S_{I,i}$ and $\delta S_{I,i}=\sum_k^{K_i}\delta S_{I,k}$ (Fig. 4a).

Likewise, for the i^{th} spatial portion of region ***II*** of volume $\delta V_{II,i}$ bounded by $\Gamma_{II,i}\cup R_{II,i}\cup\delta S_{II,i}$, (Fig. 4b), one gets accumulation rate $d\Phi_{II,i}/dt$ given as

$$\frac{d\Phi_{II,i}}{dt}=\int_{\delta V_{II,i}}\frac{\partial\eta_i}{\partial t}dV+\int_{\Gamma_{II,i}}\eta_i\boldsymbol{V}_i\bullet\boldsymbol{n}_i dA+$$
$$-\int_{R_{II,i}}\eta_i\boldsymbol{V}_{R_{II,i}}\bullet\boldsymbol{n}_i dA+\sum_k^{K_i}\int_{\delta S_{II,k}}\eta_{i\triangleright\delta S_{II,k}}\boldsymbol{U}_k\bullet\boldsymbol{n}_k dA \quad (15)$$

where $\delta S_{II,k}$ is the k^{th} portion of $\delta S_{II,i}$ and $\delta S_{II,i}=\sum_k^{K_i}\delta S_{II,k}$ (Fig. 4b). By substitution Eqs (14, 15) into Eq. (13) and letting $\delta t\to 0$ one obtains Term (2) of Eq. (6) in the rate form given as

$$\lim_{\delta t\to 0}\frac{\delta_V\Phi_{II}(\delta t)-\delta_V\Phi_I(\delta t)}{\delta t}=$$
$$=\sum_i^3\int_{P_i}\eta_i(\boldsymbol{V}_i-\boldsymbol{V}_{R_i})\bullet\boldsymbol{n}_i dA=\int_R\eta(\boldsymbol{V}-\boldsymbol{V}_{R_i})\bullet\boldsymbol{n}dA \quad (16)$$

where $R_i=R_{I,i}+R_{II,i}$ is the external boundary of the phasic volume V_i.

Term (3)

By letting $\delta t\to 0$ and taking the limit the rate form of Term (3) of Eq. (8) is

$$\lim_{\delta t\to 0}\frac{\Phi_S(t+\delta t)-\Phi_S(t)}{\delta t}=\frac{d\Phi_S(t)}{dt}=\sum_k^K\frac{d\Phi_{S_k}}{dt} \quad (17)$$

Storage $\Phi_{S_k}(t)$ in the interface of area $S_k(t)$ at an instant t is described by an integral

$$\Phi_{S_k}(t)=\int_{S_k(t)}\eta_s(y^1,y^2,t)\sqrt{a(y^1,y^2,t)}dy^1dy^2 \quad (18)$$

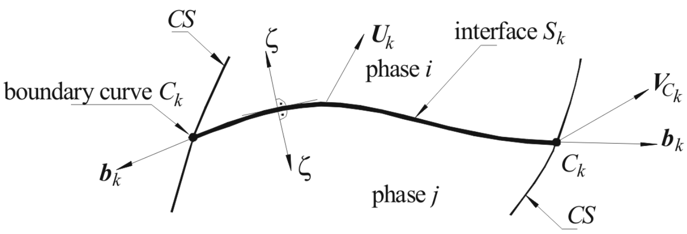


Fig. 5. Interface S_k between two phases – see description in the text

where a is the determinant of the surface metric tensor and y^1, y^2 are the surface coordinates. Hence, rate $d\Phi_{S_k}/dt$ of Eq. (17) is given now by the surface transport theorem [4]

$$\frac{d\Phi_{S_k}}{dt}=\int_{S_k}\left(\frac{\partial\eta_{s,k}}{\partial t}+\eta_{s,k}\text{div}_s\boldsymbol{U}_k\right)dA+\oint_{C_k}\eta_{s,k}(\boldsymbol{V}_{C_k}-\boldsymbol{U}_k)\bullet\boldsymbol{b}_k dl \quad (19)$$

where $dA=\sqrt{a}dy^1dy^2$ and $\text{div}_s\boldsymbol{U}_k$ is the surface divergence of $\boldsymbol{U}_k$, $\boldsymbol{V}_{C_k}$ is the spatial velocity of C_k, $\boldsymbol{b}_k$ is the unit surface vector (tangent to S_k) normal to C_k directed outward of S_k (Fig. 5).

By substitution Eq. (19) into Eq. (17), taking the limit and performing summation one gets Term (3) of Eq. (6) given by

$$\lim_{\delta t\to 0}\frac{\Phi_S(t+\delta t)-\Phi_S(t)}{\delta t}=$$
$$\int_S\left(\frac{\partial\eta_s}{\partial t}+\eta_s\text{div}_s\boldsymbol{U}\right)dA+\oint_C\eta_s(\boldsymbol{V}_C-\boldsymbol{U})\bullet\boldsymbol{b}dl \quad (20)$$

Term (4)

By letting $\delta t\to 0$ and taking the limit the rate form of the 4^{th} term of Eq. (6) is

$$\lim_{\delta t\to 0}\frac{\delta_s\Phi_{II}(\delta t)-\delta_s\Phi_I(\delta t)}{\delta t}=\lim_{\delta t\to 0}\frac{\sum_\kappa^K\delta t\frac{\Phi_{II,k}^{(s)}}{dt}-\sum_\kappa^K\delta t\frac{\Phi_{I,k}^{(s)}}{dt}}{\delta t} \quad (21)$$

Consequently accumulation rate $d\Phi_{I,k}^{(S)}/dt$ in the interface portion $\delta S_{I,k}$, (Fig. 6a), is (generalized surface transport theorem [2])

$$\frac{d\Phi_{I,k}^{(s)}}{dt}=\int_{\delta S_{I,k}}\left(\frac{\partial\eta_{s,k}}{\partial t}-\text{grad}_S\eta_{s,k}\bullet\boldsymbol{U}_k-2H_k\eta_{s,k}\boldsymbol{U}_k\bullet\zeta_k\right)dA+$$
$$+\oint_{C_{I,k}}\eta_{s,k}\boldsymbol{V}_{C_k}\bullet\boldsymbol{b}_k dl-\oint_{l_{I,k}}\eta_{s,k}\boldsymbol{V}_{s,k}\bullet\boldsymbol{b}_k dl \quad (22)$$

where $V_{s,k}$ is the spatial velocity of the surface system flowing in $\delta S_{I,k}$ and H_k is the mean curvature of $\delta S_{I,k}$. $l_{I,k}$ is the boun-

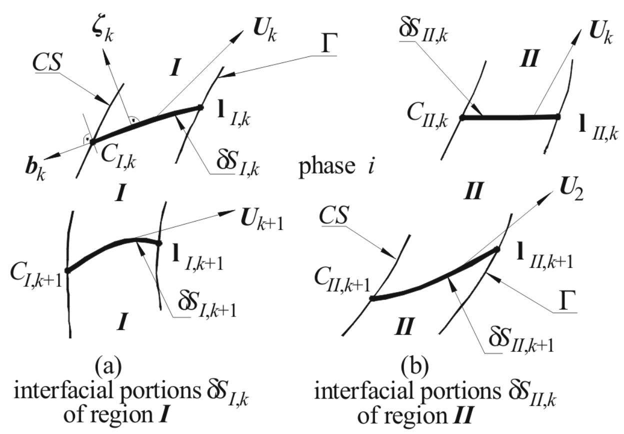


Fig. 6. Interfacial portions $\delta S_{I,k}$ and $\delta\Sigma_{II,k}$, (k = 1, 2), of region *I* and region *II*, respectively, accompanied by associated boundaries

Prosimy cytować jako: Inż. Ap. Chem. 2009, 48, 6, 168-171

dary curve formed by intersection of system boundary Γ and S_k.

Likewise, accumulation rate $d\Phi_{II,k}^{(S)}/dt$ in interface portion $\delta S_{II,k}$ located in region ***II***, (Fig. 6b), is

$$\frac{d\Phi_{II,k}^{(S)}}{dt} = \int_{\delta S_{II,k}} \left(\frac{\partial \eta_{s,k}}{\partial t} - \mathrm{grad}_s \eta_{s,k} \bullet \boldsymbol{U_k} - 2H_k \eta_{s,k} \boldsymbol{U_k} \bullet \zeta_k \right) dA - \\ - \oint_{C_{II,k}} \eta_{s,k} \boldsymbol{V}_{C_k} \bullet \boldsymbol{b}_k dl + \oint_{l_{II,k}} \eta_{s,k} \boldsymbol{V}_{s,k} \bullet \boldsymbol{b}_k dl \qquad (23)$$

By substitution Eqs (22, 23) into Eq. (21) and letting $\delta t \to 0$ one finds Term (4) of Eq. (6) becomes to be expressed in the rate form as

$$\lim_{\delta t \to 0} \frac{\delta_s \Phi_{II}(\delta t) - \delta_s \Phi_I(\delta t)}{\delta t} = \\ = \Sigma_k^K \oint_{C_{II,k}} \eta_{s,k} (\boldsymbol{V}_{s,k} - \boldsymbol{V}_{C_k}) \bullet \boldsymbol{b}_k dl = \oint_C \eta_s (\boldsymbol{V}_s - \boldsymbol{V}_C) \bullet \boldsymbol{b} dl \qquad (24)$$

By substitution expression (7) onto left side of Eq. (6) and relations (12), (16), (20) and (24) into the right side of Eq. (6) we can generalize the *RTT* for three phase systems as follows

$$\underbrace{\frac{d\Phi_\Sigma}{dt}}_{\text{accumulation in }\Sigma} = \underbrace{\int_V \frac{\partial \eta}{\delta t} dV + \int_V \mathrm{div}(\eta \boldsymbol{V}_\Re) dV + \int_R \eta (\boldsymbol{V} - \boldsymbol{V}_R) \bullet \boldsymbol{n} dA}_{\text{phasic terms}} + \qquad (25)$$

$$+ \underbrace{\int_S \left(\frac{\partial \eta_S}{\partial t} + \eta_S \mathrm{div}_S \boldsymbol{U} \right) dA + \oint_C \eta_s (\boldsymbol{V}_C - \boldsymbol{U}) \bullet \boldsymbol{b} dl + \oint_C \eta_s (\boldsymbol{V}_s - \boldsymbol{V}_C) \bullet \boldsymbol{b} dl}_{\text{interfacial terms}}$$

The third term on the right side of Eq. (25) can be modified by the use of the *Gauss's* theorem for spatial domains [3]. Thus one gets

$$\int_R \eta (\boldsymbol{V} - \boldsymbol{V}_R) \bullet \boldsymbol{n} dA = \\ = \int_V \mathrm{div}[\eta (\boldsymbol{V} - \boldsymbol{V}_{\Re_i})] dV + \int_S \| \eta_{\rhd S} (\boldsymbol{V}_{\rhd S} - \boldsymbol{U}) \bullet \zeta \| dA \qquad (26)$$

where

$$\int_S \| \eta_{\rhd S} (\boldsymbol{V}_{\rhd S} - \boldsymbol{U}) \bullet \zeta \| dA = \\ = \Sigma_k^K \int_{S_k} \left\| \eta_{\rhd S_k}^+ (\boldsymbol{V}_{\rhd S_k}^+ - \boldsymbol{U_k}) \bullet \zeta^+ + \eta_{\rhd S_k}^- (\boldsymbol{V}_{\rhd S_k}^- - \boldsymbol{U_k}) \bullet \zeta^- \right\| dA \qquad (27)$$

Substitution of Eq. (26) into Eq. (25) yields the final form of the *RTT* attempted as follows

$$\underbrace{\frac{d\Phi_\Sigma}{dt}}_{\text{accumulation in }\Sigma} = \underbrace{\int_V \frac{\partial \eta}{\partial t} dV + \int_V \mathrm{div}(\eta \boldsymbol{V}_\Re) dV}_{\text{accumulation in moving spatial phasic domains of volume } V} + \underbrace{\int_V \mathrm{div}[\eta (\boldsymbol{V} - \boldsymbol{V}_\Re)] dV}_{\text{transport of EQ by movement of spatial subsystems across }\Re} +$$

$$+ \underbrace{\int_S \left(\frac{\partial \eta_s}{\partial t} + \eta_s \mathrm{div}_s \boldsymbol{U} \right) dA + \oint_C \eta_s (\boldsymbol{V}_C - \boldsymbol{U}) \bullet \boldsymbol{b} dl}_{\text{accumulation of interfacial domains of area } S} +$$

$$+ \underbrace{\oint_C \eta_s (\boldsymbol{V}_s - \boldsymbol{V}_C) \bullet \boldsymbol{b} dl}_{\text{transport of EQ by movement of surface subsystems across bounding curves } C} + \underbrace{\int_S \| \eta_{\rhd S} (\boldsymbol{V}_{\rhd S} - \boldsymbol{U}) \bullet \zeta \| dA}_{\text{transport between interface } S \text{ and bulk phases}} \qquad (28)$$

In this notation subscript $\rhd S_k$ stands for value of a phasic property referenced at infinitesimally close spatial position to S_k. Note also in Eq. (28) the spatial divergence operator div and surface divergence operator div_s are different, see formulas for these operators given by *Slattery* [2].

Concluding remarks

The *RTT* is a basic tool in development of the local instantaneous model equations together with corresponding jump conditions based on which averaged models can be derived. The form of *RTT* given by Eq. (28) is the most general because it expresses the rate of accumulation in a three phase system in terms of moving and deformable *CV* of arbitrary prescribed configuration in which *EQ* can be accumulated both by the phases and interface. Worthy to mention is applicability of *RTT* relation developed also to multiphase systems provided that particular terms can account in contributions done by all spatial and surface subsystems involved.

REFERENCES

1. *C. Truesdell and R. A. Toupin*: The classical field theories, in S. Flügge (ed.), Handbuch der Physik, vol.**1/3**, Berlin, Springer-Verlag, 1960.
2. *J. C. Slattery*: Interfacial Transport Phenomena, Springer-Verlag, New York Inc., 1990.
3. *W. Kaplan*: Advanced Calculus, 2nd edition, Addison-Wesley, Reading, 1973.
4. *R. Aris*: Vectors, Tensors, and the Basic Equations of Fluid Mechanics, Prentice-Hall, Inc., 1962.